A SOLVABILITY CRITERION FOR THE LIE ALGEBRA OF DERIVATIONS OF A FAT POINT

MATHIAS SCHULZE

Abstract. We consider the Lie algebra of derivations of a zero dimensional local complex algebra. We describe an inequality involving the embedding dimension, the order, and the first deviation that forces this Lie algebra to be solvable. Our result was motivated by and generalizes the solvability of the Yau algebra of an isolated hypersurface singularity.

1. Introduction

Let $f \in \mathfrak{m} = \langle x_1, \ldots, x_n \rangle \subseteq \mathbb{C}\{x_1, \ldots, x_n\} = \mathscr{O}$ define an isolated hypersurface singularity $X = \{f = 0\} \subseteq (\mathbb{C}^n, 0)$. The *Tyurina algebra* of X is the finite $\mathbb{C}$-algebra $A(X) = \mathscr{O}/\langle f, J(f)\rangle$ where $J(f) = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle$. By a result of Mather and Yau [MY82], $A(X)$ determines the analytic isomorphism class of X. The *Yau algebra* $L(X) = \operatorname{Der}_{\mathbb{C}} A(X)$ is the Lie algebra of derivations of $A(X)$. Its structure and the interplay of its properties with those of X do not seem to be understood in general. For instance, there is the *recognition problem* asking what Lie algebras can arise as $L(X)$, the *recognition problem* asking what information on X can be restored from $L(X)$, and in particular the *classification problem* asking up to what extend $L(X)$ determines the isomorphism class of $A(X)$ and hence of X. In the case of simple singularities, the classification problem has been studied by Elashvili and Khimshiashvili [EK06].

An important result on the recognition problem was formulated by Yau [Yau91, Thm. 2]: $L(X)$ is a solvable complex Lie algebra. The purpose of this note is to prove the following generalization of this result in which we replace J by a general zero dimensional ideal. Our approach is inspired by the result of Müller [Mül86, Hilfssatz 2] that any ideal of an analytic algebra invariant under a reductive algebraic group is minimally generated by an invariant vector space.

Theorem 1. *Let S be a zero-dimensional local $\mathbb{C}$-algebra of embedding dimension* $\operatorname{embdim}(S)$ *and order* $\operatorname{ord}(S)$*, and denote by $\varepsilon_1(S)$ its first deviation. Then the Lie algebra* $\operatorname{Der}_{\mathbb{C}} S$ *is solvable if* $\varepsilon_1(S) + 1 < \operatorname{embdim}(S) + \operatorname{ord}(S)$.

Recall that, by definition, $\varepsilon_1(S) = \dim_{\mathbb{C}} H_1(S)$ where $H_\bullet(S)$ is the Koszul algebra of S. More explicitly, Theorem 1 applies to $S = R/I$ where $R = \mathbb{C}[\![x_1, \ldots, x_n]\!]$ and $I \subseteq R$ is a zero dimensional ideal with $I \subseteq \mathfrak{m}^m$ where $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ and $m \geq$

Date: September 9, 2009.
1991 *Mathematics Subject Classification.* 32S10, 17B30.
Key words and phrases. fat point, derivation, Lie algebra, solvability, hypersurface singularity, Yau algebra.
The author was supported by the College of Arts & Sciences at Oklahoma State University through a FY 2010 ASR+1 award.

2 is chosen maximal. Then $n = \operatorname{embdim}(S)$, $m = \operatorname{ord}(S)$, and $\varepsilon_1(S) = \dim_{\mathbb{C}}(I/\mathfrak{m}I)$ is the minimal number of generators of I [BH93, Thm. 2.3.2.(b)].

By [BH93, Thm. 2.3.3.(b)], S being a complete intersection is equivalent to $\varepsilon_1(S) = \operatorname{embdim}(S) - \dim(S)$. In this case, the inequality in Theorem 1 reduces to $1 < \operatorname{ord}(S)$ which holds trivially unless $S = \mathbb{C}$.

Corollary 2. *If S is a zero-dimensional complex complete intersection then $\operatorname{Der}_{\mathbb{C}} S$ is a solvable Lie algebra.*

This result applies in particular to the *Milnor algebra* $S = \mathscr{O}/J(f)$ of a a function $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point.

Corollary 3. *If S is a Milnor algebra then $\operatorname{Der}_{\mathbb{C}} S$ is a solvable Lie algebra.*

Using a theorem of Kempf [Kem93, Thm. 13] in the order 3 case, also the solvability of Yau algebras becomes a corollary of Theorem 1.

Corollary 4 (Yau's solvability theorem)**.** *The Yau algebra of any isolated hypersurface singularity is a solvable Lie algebra.*

Remark 5. We do not understand the "induction step" in Yau's proof of Corollary 4 [Yau91, Thm. 2]: Yau considers the Taylor expansion $f = \sum_{i=k+1}^{\infty} f_i$ of f and assumes $A(f)$ to be $\mathfrak{s}$-invariant for some $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{s} \subseteq \mathfrak{gl}_n(\mathbb{C})$. By hypothesis, $J(f)$ contains $\mathfrak{m}^m$ for some $m \geq k$ and, by finite determinacy, one can assume that f is a polynomial of degree $d \leq m$. Then also the larger ideal $J(f_{k+1}) + \cdots + J(f_d)$ of homogeneous parts of $J(f)$ contains $\mathfrak{m}^m$ and hence the quotient $(\mathfrak{m}^\ell + J(f_{k+1}) + \cdots + J(f_\ell))/(\mathfrak{m}^{\ell+1} + J(f_{k+1}) + \cdots + J(f_\ell))$ is zero for any $\ell \geq m$. However, Yau identifies a subspace J_ℓ of this zero space with the vector space $\langle \partial f_{\ell+1}/\partial x_1, \ldots, \partial f_{\ell+1}/\partial x_n \rangle_{\mathbb{C}}$ in order to conclude that the latter is $\mathfrak{s}$-invariant. We do not see any reason why this space should be $\mathfrak{s}$-invariant in general for $\ell > k$.

However, for $\ell = k$, this invariance holds true and Yau's argument [Yau91, Thm. 1] using Kempf's result [Kem93, Thm. 13] proves Corollary 4 in the homogeneous case (cf. Remark 8). We shall use the same idea to prove the order 3 case in general (cf. Remark 9). In the higher order cases, we shall apply our main result Theorem 1.

2. Proofs

For an ideal J in an analytic algebra S, denote by $\operatorname{Der}_J S \subseteq \operatorname{Der}_{\mathbb{C}} S$ the Lie subalgebra of all $\delta \in \operatorname{Der}_{\mathbb{C}} S$ for which $\delta(J) \subseteq J$. We shall first reformulate the claim of Theorem 1 in terms of the Lie algebra $\operatorname{Der}_{\bar{I}} \bar{R}$ where $\bar{R} = R/\mathfrak{m}^{\ell+1}$, for sufficiently large $\ell \geq 2$, and $\bar{I}$ is the image of I in $\bar{R}$. To this end, we shall use the following result.

Lemma 6. *For J be an ideal in $R = \mathbb{C}[\![x_1, \ldots, x_n]\!]$. Then there is a natural isomorphism of Lie algebras*

$$(\operatorname{Der}_J R)/(J \cdot \operatorname{Der}_{\mathbb{C}} R) \cong \operatorname{Der}_{\mathbb{C}}(R/J).$$

Proof. By definition, there is a map $\varphi\colon \operatorname{Der}_J R \to \operatorname{Der}_{\mathbb{C}}(R/J)$ whose kernel contains $J \cdot \operatorname{Der}_{\mathbb{C}} R$. Note that $\operatorname{Der}_{\mathbb{C}} R$ is a free R-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and that the coefficient of $\partial/\partial x_i$ in $\delta \in \operatorname{Der}_{\mathbb{C}} R$ is $\delta(x_i)$. So if $\delta \in \ker \varphi$, then $\delta(x_i) \in J$ and hence $\delta \in J \cdot \operatorname{Der}_{\mathbb{C}} R$. This proves injectivity. By a result of Scheja and Wiebe [SW73, (2.1)], any $\bar{\delta} \in \operatorname{Der}_{\mathbb{C}}(R/J)$ lifts to a $\delta \in \operatorname{Der}_{\mathbb{C}} R$ which is then necessarily in $\operatorname{Der}_J R$. This proves surjectivity and the claim follows. $\square$

Consider now the situation of Theorem 1. We shall use the notation introduced in the paragraph following Theorem 1. Since I is $\mathfrak{m}$-primary by hypothesis,

$$\mathfrak{m}^\ell \subseteq I, \text{ for some } \ell \geq 2, \tag{1}$$

and we set

$$\bar{I} := I/\mathfrak{m}^{\ell+1} \subseteq \bar{\mathfrak{m}} := \mathfrak{m}/\mathfrak{m}^{\ell+1} \subseteq \bar{R} := R/\mathfrak{m}^{\ell+1}. \tag{2}$$

Note that

$$\bar{I}/(\bar{\mathfrak{m}} \cdot \bar{I}) \cong I/(\mathfrak{m} \cdot I) \tag{3}$$

and hence $\bar{I}$ has the same minimal number of generators as I. As $\mathfrak{m}$ is an associated prime of I, it follows from a result of Scheja and Wiebe [SW73, (2.5)] that

$$\operatorname{Der}_I R \subseteq \operatorname{Der}_{\mathfrak{m}} R. \tag{4}$$

Using the Leibniz rule and [SW73, (2.5)] again, one shows that

$$\operatorname{Der}_{\mathfrak{m}} R = \operatorname{Der}_{\mathfrak{m}^i} R, \text{ for all } i \geq 2. \tag{5}$$

Using [Jac79, Ch. I, §7, Lem.], the following result reduces the claim of Theorem 1 to prove solvability of $\operatorname{Der}_{\bar{I}} \bar{R}$.

Lemma 7. $\operatorname{Der}_{\mathbb{C}}(S)$ *a subquotient of* $\operatorname{Der}_{\bar{I}} \bar{R}$.

Proof. Using (5), Lemma 6 yields a natural isomorphism

$$(\operatorname{Der}_{\mathfrak{m}} R)/(\mathfrak{m}^{\ell+1} \cdot \operatorname{Der}_{\mathbb{C}} R) \cong \operatorname{Der}_{\mathbb{C}} \bar{R}. \tag{6}$$

By the choice of ℓ and (4),

$$\mathfrak{m}^{\ell+1} \cdot \operatorname{Der}_{\mathbb{C}} R \subseteq \mathfrak{m}^{\ell} \cdot \operatorname{Der}_{\mathbb{C}} R \subseteq \operatorname{Der}_I R \subseteq \operatorname{Der}_{\mathfrak{m}} R$$

and hence (6) induces an injection

$$(\operatorname{Der}_I R)/(\mathfrak{m}^{\ell+1} \cdot \operatorname{Der}_{\mathbb{C}} R) \hookrightarrow \operatorname{Der}_{\bar{I}} \bar{R}. \tag{7}$$

Lemma 6 also yields an isomorphism

$$(\operatorname{Der}_I R)/(I \cdot \operatorname{Der}_{\mathbb{C}} R) \cong \operatorname{Der}_{\mathbb{C}}(R/I). \tag{8}$$

Combining (7) and (8) makes $\operatorname{Der}_{\mathbb{C}}(R/I)$ a subquotient of $\operatorname{Der}_{\bar{I}} \bar{R}$ as claimed. □

In order to apply the reduction given by Lemma 7, we shall need (4) and (5) also for R replaced by $\bar{R}$. By [SW73, (2.5)], any $\delta \in \operatorname{Der}_{\bar{I}} \bar{R}$ lifts to a $\delta' \in \operatorname{Der}_{\mathbb{C}} R$. By (1), (2), and (4), $\delta' \in \operatorname{Der}_I R = \operatorname{Der}_{\mathfrak{m}} R$ and hence $\delta \in \operatorname{Der}_{\bar{\mathfrak{m}}} \bar{R}$. Thus,

$$\operatorname{Der}_{\bar{I}} \bar{R} \subseteq \operatorname{Der}_{\bar{\mathfrak{m}}} \bar{R} \tag{9}$$

analogous to (4), and, using the the Leibniz rule, one deduces

$$\operatorname{Der}_{\bar{I}} \bar{R} \subseteq \operatorname{Der}_{\bar{\mathfrak{m}} \cdot \bar{I}} \bar{R}. \tag{10}$$

The analogue of (5) is proved similarly and reads

$$\operatorname{Der}_{\mathbb{C}} \bar{R} = \operatorname{Der}_{\bar{\mathfrak{m}}^i} \bar{R}, \text{ for all } i \geq 1. \tag{11}$$

After these preparations we are ready to finish the

Proof of Theorem 1. Assume that $\operatorname{Der}_{\mathbb{C}} S$ is not solvable which implies, by Lemma 7, that $\operatorname{Der}_{\bar{I}} \bar{R}$ is not solvable. By Levi's theorem [Jac79, Ch. III, §9, Levi's thm.] and the structure of semisimple Lie algebras [Jac79, Ch. IV, §3], there is a Lie subalgebra

$$\mathfrak{sl}_2(\mathbb{C}) \cong \langle H, X, Y\rangle = \mathfrak{s} \subseteq \operatorname{Der}_{\bar{I}} \bar{R}$$

where H, X, Y are standard generators subject to the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

By (9), (11) for $i = 2$, and complete reducibility [Jac79, Ch. III, §7, Thm. 8], we can assume that the $\mathbb{C}$-vector space $\bar{V} = \langle \bar{x}_1, \ldots, \bar{x}_n\rangle_{\mathbb{C}}$ spanned by the $\bar{x}_i = x_i + \mathfrak{m}^{\ell+1} \in \bar{R}$ is an $\mathfrak{s}$-invariant vector space. This means that the representation of the Lie algebra $\mathfrak{s}$ on $\bar{R}$ is linear with respect to the coordinates $\bar{x}_1, \ldots, \bar{x}_n$. By the same argument using (10) instead of (5) for $i = 2$, we can find an $\mathfrak{s}$-invariant vector space $\bar{F} \subseteq \bar{I}$ that maps isomorphically onto $\bar{I}/(\bar{\mathfrak{m}} \cdot \bar{I})$. In other words, a basis of $\bar{F}$ is a minimal set of generators of $\bar{I}$. Then, by (3) and according to the interpretation following Theorem 1, we have to show that

$$\dim_{\mathbb{C}} \bar{F} \geq m + n - 1. \tag{12}$$

Now, again by complete reducibility and by the classification of irreducible modules [Jac79, Ch. VII, §3], $\bar{V}$ decomposes into irreducible $\mathfrak{s}$-modules each of which is generated by a highest weight vector, which means an H-homogeneous vector in $\ker X$. We may assume that $\bar{x}_1, \ldots, \bar{x}_k$ are these highest weight vectors, that is, $X(x_i) = 0$ for $i = 1, \ldots, k$, and that their H-weights form a nonincreasing sequence, that is, $H(x_1)/x_1 \geq \cdots \geq H(x_k)/x_k$. Moreover, we can choose the remaining variables $\bar{x}_{k+1}, \ldots, \bar{x}_n$ to be H-homogeneous and compatible with the decomposition of $\bar{V}$ into irreducible $\mathfrak{s}$-modules, that is, X and Y map variables to multiples of variables. Since I is zero dimensional,

$$(I + \langle x_{i+1}, \ldots, x_n\rangle)/\langle x_{i+1}, \ldots, x_n\rangle \subseteq R/\langle x_{i+1}, \ldots, x_n\rangle \cong \mathbb{C}[\![x_1, \ldots, x_i]\!]$$

is a zero dimensional ideal in an i-dimensional power series ring. So it must have at least i generators. By the analogue of (3) with R replaced by $R/\langle x_{i+1}, \ldots, x_n\rangle$, the same holds for $(\bar{I} + \langle \bar{x}_{i+1}, \ldots, \bar{x}_n\rangle)/\langle \bar{x}_{i+1}, \ldots, \bar{x}_n\rangle$ which is generated by the image of $\bar{F}$ in $\bar{R}/\langle \bar{x}_{i+1}, \ldots, \bar{x}_n\rangle$. Therefore we can associate to each $i = 1, \ldots, k$ an H-homogeneous element $\bar{f}_i \in \bar{F}$ such that $\bar{f}_1, \ldots, \bar{f}_i$ are linearly independent modulo $\langle \bar{x}_{i+1}, \ldots, \bar{x}_n\rangle$. These elements generate an H-homogeneous k-dimensional vector space

$$\bar{F}' = \langle \bar{f}_1, \ldots, \bar{f}_k\rangle_{\mathbb{C}}.$$

As $\bar{x}_1, \ldots, \bar{x}_k$ are highest weight vectors in $\bar{V}$ and hence in $\bar{R}$, the subring $\mathbb{C}[\bar{x}_1, \ldots, \bar{x}_k] \subseteq \bar{R}$ consists of highest weight vectors by the Leibnitz rule. Thus,

$$\bar{g}_i(\bar{x}_1, \ldots, \bar{x}_k) = \bar{f}_i(\bar{x}_1, \ldots, \bar{x}_k, 0, \ldots, 0), \text{ for } i = 1, \ldots, k, \tag{13}$$

are linearly independent highest weight vectors in $\bar{R}$ and

$$\bar{G} = \langle \bar{g}_1, \ldots, \bar{g}_k\rangle_{\mathbb{C}}$$

is an H-homogeneous vector space. Any highest weight vector outside of $\mathbb{C}[\bar{x}_1, \ldots, \bar{x}_k]$ can be chosen to lie in the ideal $\langle \bar{x}_{k+1}, \ldots, \bar{x}_n\rangle \subseteq \bar{R}$. As the latter is stable by the Borel algebra $\langle H, Y\rangle_{\mathbb{C}}$, it contains also the $\mathfrak{s}$-module generated by such a vector. This shows that there is a projection of $\mathfrak{s}$-modules

$$\bar{F} \supseteq \mathfrak{s} \cdot \bar{F}' \twoheadrightarrow \mathfrak{s} \cdot \bar{G}.$$

So in order to find a lower bound for $\dim_{\mathbb{C}} \bar{F}$ such as (12), we may as well assume that $\bar{f}_i = \bar{g}_i$ for $i = 1, \dots, k$.

Recall the hypothesis $I \subseteq \mathfrak{m}^m$, $m \geq 2$, which implies $\bar{I} \subseteq \bar{\mathfrak{m}}^m$, and denote by m_i the H-weight of $\bar{x}_i$. Then $\bar{g}_i$ has H-weight at least $m \cdot m_i$ and hence $\mathfrak{s} \cdot \bar{g}_i$ has dimension at least $m \cdot m_i + 1$. Note that the dimension of $\bar{V}$ is $n = m_1 + 1 + \cdots + m_k + 1$. It follows from the preceding arguments that

$$\dim_{\mathbb{C}} \bar{F} \geq (m-1) \cdot (m_1 + \cdots + m_k) + n. \tag{14}$$

As $\mathfrak{s} \subseteq \mathfrak{gl}_n(\mathbb{C})$, at least one of the irreducible $\mathfrak{s}$-modules in $\bar{V}$ must be nontrivial which means that $m_i \geq 1$ for some $i \in \{1, \dots, k\}$. It follows that $\dim_{\mathbb{C}} \bar{F} \geq m+n-1$ as claimed. This finishes the proof of Theorem 1. $\square$

Remark 8. In the setting of reductive groups of automorphisms of analytic algebras, an invariant set of minimal generators such as $\bar{F}$ in the proof of Theorem 1 has been constructed by Müller [Mül86, Hilfssatz 2].

Finally we give the

Proof of Corollary 4. By a remark of Yau [Yau91, §2 Rem.], one can reduce to the case where $f \in \mathfrak{m}^3$. Theorem 1 applied to $I = \langle f, J(f) \rangle$ yields the claim

- if $\operatorname{ord}(X) \geq 4$,
- if X is quasihomogeneous and hence $I = J(f)$, or
- if $\bar{V}$ contains a 3-dimensional $\mathfrak{s}$-module using (14).

So we may assume that $f \in \mathfrak{m}^3 \setminus \mathfrak{m}^4$ and that $\bar{V}$ contains exactly one 2-dimensional and $n-2$ many 1-dimensional irreducible $\mathfrak{s}$-modules, say $\langle \bar{x}_1, \bar{x}_n \rangle_{\mathbb{C}}$ and $\langle \bar{x}_2 \rangle_{\mathbb{C}}, \dots, \langle \bar{x}_{n-1} \rangle_{\mathbb{C}}$. In particular, the H-weights of $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}, \bar{x}_n$ equal $m_1 = 1, m_2 = 0, \dots, m_k = 0, -1$. Denote by $\bar{f}^3 \in \bar{V}^3$ the homogeneous part of degree 3 of $\bar{f} = f + \mathfrak{m}^{\ell+1} \in \bar{R}$ with respect to $\bar{x}_1, \dots, \bar{x}_n$. Then $J(\bar{f}^3) \subseteq \bar{V}^2$ is an $\mathfrak{s}$-module. By a result of Kempf [Kem93, Thm. 13], there is an $\mathfrak{s}$-invariant polynomial $\bar{g} \in \bar{V}^3$ such that $J(\bar{g}) = J(\bar{f}^3)$. In particular, $\bar{g}$ must be H-homogeneous of weight 0 and hence in the span of $x_1 \cdot \langle x_2, \dots, x_{n-1} \rangle_{\mathbb{C}} \cdot x_n$ and $\langle x_2, \dots, x_{n-1} \rangle_{\mathbb{C}}^3$. But then $X(\bar{g}) = 0 = Y(\bar{g})$ forces $\bar{g}$ and hence $J(\bar{f}^3)$ to be independent of $\bar{x}_1$ and $\bar{x}_n$. This shows that $\bar{g}_1$ in (13) involves at least a third power of $\bar{x}_1$. Thus, the inequality (14) can be improved by setting $m = 3$ which suffices to conclude Corollary 4. $\square$

Remark 9. We do not know how to avoid the rather deep result of Kempf in the proof of the order 3 case in Corollary 4.

References

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020)

[EK06] A. Elashvili and G. Khimshiashvili, *Lie algebras of simple hypersurface singularities*, J. Lie Theory **16** (2006), no. 4, 621–649. MR MR2270654

[Jac79] Nathan Jacobson, *Lie algebras*, Dover Publications Inc., New York, 1979, Republication of the 1962 original. MR MR559927 (80k:17001)

[Kem93] George R. Kempf, *Jacobians and invariants*, Invent. Math. **112** (1993), no. 2, 315–321. MR MR1213105 (94b:14043)

[Mül86] Gerd Müller, *Reduktive Automorphismengruppen analytischer* **C**-*Algebren*, J. Reine Angew. Math. **364** (1986), 26–34. MR MR817635 (88d:32041)

[MY82] John N. Mather and Stephen S. T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), no. 2, 243–251. MR MR674404 (84c:32007)

[SW73] Günter Scheja and Hartmut Wiebe, *Über Derivationen von lokalen analytischen Algebren*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), Academic Press, London, 1973, pp. 161–192. MR MR0338461 (49 #3225)

[Yau91] Stephen S.-T. Yau, *Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by* $\mathrm{sl}(2, \mathbf{C})$ *invariant polynomials*, Amer. J. Math. **113** (1991), no. 5, 773–778. MR MR1129292 (92j:32125)

M. Schulze, Oklahoma State University, 401 Mathematical Sciences, Stillwater, OK 74078, USA

E-mail address: mschulze@math.okstate.edu